Abstract

In this paper we have defined the sign functions $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\varepsilon_4$, $\varepsilon_5$ and the vector fields $X_u$, $X_v$, $n_u$ and $n_v$ which have taken derivatives with $(u,v)$ parameters of the tangent vector field $X$ of any surface in Lorentz space and we get fundamental forms, Weingarten equations, Olin-Rodrigues and Gauss formulae. Beside these we calculate Gauss and mean curvatures.

Keywords: Lorenz Surface, Fundamental Forms, Curvatures, Weingarten Formulae

Preliminaries

It is well known that in a Lorentzian Manifold we can find three types of submanifolds: Space-like (or Riemannian), time-like (Lorentzian) and light-like (degenerate or null), depending on the induced metric in the tangent vector space. Lorentz surfaces has been examined in numerous articles and books. In this article, however, we have examined some characteristics belonging to the surface by making some special choices on tangent space along the coordinate curves of the surface. Let $\mathbb{R}^3$ be endowed with the pseudoscalar product of $X$ and $Y$ is defined by

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3 \quad X = (x_1, x_2, x_3), \; Y = (y_1, y_2, y_3)$$

$\langle \mathbb{R}^3, \langle \rangle \rangle$ is called 3-dimensional Lorentzian space denoted by $L^3$ [1]. The Lorentzian vector product is defined by

$$X \times Y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

A vector fields $X$ in $L^3$ is called a space-like, light-like, time-like vector field if $\langle X, X \rangle > 0$, $\langle X, X \rangle = 0$ or $\langle X, X \rangle < 0$ accordingly. For $X \in L^3$, the norm of $X$ defined by

$$\|X\| = \sqrt{\langle X, X \rangle}$$

and $X$ is called a unit vector if $\|X\| = 1$[2].
1. INTRODUCTION

**Definition 1.1.** A symmetric bilinear form $b$ on vector space $V$ is

i) positive [negative] definite provided $v \neq 0$ implies $b(v, v) > 0 [\leq 0]$

ii) positive [negative] semi-definite provided $v \geq 0 [v \leq 0]$ for all $v \in V$

iii) non-degenerate provided $b(v, w) = 0$ for all $w \in V$ implies $v = 0 [1]$.

**Definition 1.2.** A scalar product $g$ on a vector space $V$ is a non-degenerate symmetric bilinear form on $V [1]$.

**Definition 1.3.** The index $\nu$ of the symmetric bilinear form $b$ on $V$ is the largest integer that is the dimension of a subspace $W \subset V$ on which $\exists b_W$ is negative definite $[1]$.

**Lemma 1.4.** A scalar product space $V \neq 0$ has an orthonormal basis for $V$, $\varepsilon_i = \langle \varepsilon_i, \varepsilon_i \rangle$. Then each $\alpha \in V$ has a unique expression $[1]$,

$$\alpha = \sum_{i=1}^{n} \varepsilon_i \langle \varepsilon_i, \varepsilon_i \rangle \varepsilon_i$$

**Lemma 1.5.** For any orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ for $V$, the number of negative signs in the signature $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is the index $\nu$ of $V$ $[1]$.

**Definition 1.6.** A metric tensor $g$ on a smooth manifold $M$ is a symmetric nondegenerate $(0, 2)$ tensor field on $M$ of constant index $[1]$.

**Definition 1.7.** A semi-Riemannian manifold is a smooth manifold furnished with a metric tensor $g$.

**Definition 1.8.** A semi-Riemannian submanifold $M$ with $(n-1)$-dimensional of a semi-Riemannian manifold $M$ with $n$-dimensional is called semi-Riemannian hypersurface of $M [1]$.

2. FUNDAMENTAL FORMS

Let us denote space-like or time-like surface in $L^3$ as $M$ and let the equation of $M$ be $\vec{X}(u, v) = \vec{X}$ with the parameter $t (t \in \mathbb{R})$. $X_u$ and $X_v$ are the tangent vector fields along coordinate curves on $M$ and at any point these vector fields can be described with the parameter $t$ of the coordinate curve $X$ respectively. The velocity vector of this curve at any point $p(u, v)$ is,

$$X'(t) = \frac{dX}{dt} = \frac{\partial X}{\partial u} du + \frac{\partial X}{\partial v} dv = X_u du + X_v dv$$

and it is perpendicular to $X_u \times X_v$. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ be sign functions and $X_u, X_v, n$ tangent and normal vector fields on $M$ and so we define the following equalities.

$$\langle n_u, n_u \rangle = \varepsilon_1 \|n_u\|^2, \quad \langle n_v, n_v \rangle = \varepsilon_2 \|n_v\|^2, \quad \langle n, n \rangle = \varepsilon_3 \|n\|^2$$

$$\langle X_u, X_u \rangle = \varepsilon_3 \|X_u\|^2, \quad \langle X_v, X_v \rangle = \varepsilon_4 \|X_v\|^2$$

$$\langle n_v, X_v \rangle = \frac{M}{\sqrt{\varepsilon_2 \varepsilon_3}}, \quad \langle n_v, X_v \rangle = \frac{N}{\sqrt{\varepsilon_2 \varepsilon_4}}, \quad \langle X_u, X_v \rangle = \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}, \quad \langle n_u, X_v \rangle = \frac{M}{\sqrt{\varepsilon_1 \varepsilon_4}}$$

$$\langle X_u, X_v \rangle = \varepsilon_3 E, \quad \langle X_v, X_v \rangle = \varepsilon_4 G, \quad \langle n_u, X_u \rangle = \frac{L}{\sqrt{\varepsilon_1 \varepsilon_3}}, \quad H = \varepsilon_3 \varepsilon_4 GE - \frac{F^2}{\varepsilon_3 \varepsilon_4}$$

Hence arc length of the curve $X(t)$ is defined by,

$$s = \int \|X'(t)\| dt = \varepsilon_3 Edu^2 + \frac{2F}{\varepsilon_3 G} dv + \varepsilon_4 Gdv^2$$

Equation [2.1] is called F.F.F. quadratic form of $M$ and we can define it as

$$I = \varepsilon_3 Edu^2 + \frac{2F}{\varepsilon_3 G} dv + \varepsilon_4 Gdv^2$$

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Let the point \( Q(u + \Delta u, v + \Delta v) \) be any near-by point at neighbourhood of \( p(u, v) \) on the surface which belongs to the set \( C^2 \). Distance between tangent plane at \( p(u, v) \) and at any point \( Q \) is \( d = \langle \overline{n}, \overline{PQ} \rangle \) and using Taylor formula, then we can write,

\[
\overline{PQ} = \Delta X = \Delta u X_u + \Delta v X_v + \frac{1}{2} \left( \Delta u^2 X_{uu} + 2 \Delta u \Delta v X_{uv} + \Delta v^2 X_{vv} + \varepsilon \right)
\]

and we obtain,

\[
d = \langle \overline{n}, \overline{\Delta X} \rangle = \frac{1}{2} \left( \Delta u^2 \langle n, X_{uu} \rangle + 2 \Delta u \Delta v \langle n, X_{uv} \rangle + \Delta v^2 \langle n, X_{vv} \rangle + \langle n, \varepsilon \rangle \right)
\]

For \( (\Delta u, \Delta v) \rightarrow (0, 0) \) we obtain,

\[
\lim_{(\Delta u, \Delta v) \rightarrow (0, 0)} \langle n, \varepsilon \rangle = 0
\]

so \( \langle n, \varepsilon \rangle \) has no role to define the sign of \( d \). For \( Q \) points which are sufficiently close to \( P \), we can write,

\[
d = \frac{1}{2} \left( \langle n, X_{uu} \rangle du^2 + 2 \langle n, X_{uv} \rangle dudv + \langle n, X_{vv} \rangle dv^2 \right)
\]

Furthermore, \( X_u \) and \( X_v \) vectors are normal to \( n \) so \( \langle n, X_u \rangle \) and \( \langle n, X_v \rangle \) are equal to zero. Let us take derivative \( \langle n, X_u \rangle \) and \( \langle n, X_v \rangle \) with respect to \( u \) and \( v \);

\[
\langle n, X_u \rangle + \langle n, X_{uu} \rangle = 0 \quad \Rightarrow \quad -\langle n, X_u \rangle = \langle n, X_{uu} \rangle = \frac{L}{\sqrt{\varepsilon_1 \varepsilon_3}}
\]

\[
\langle n, X_v \rangle + \langle n, X_{vv} \rangle = 0 \quad \Rightarrow \quad -\langle n, X_v \rangle = \langle n, X_{vv} \rangle = \frac{N}{\sqrt{\varepsilon_2 \varepsilon_4}}
\]

and use the formulae \( II = \langle n, d^2 X \rangle \) and \( \langle n, d^2 X \rangle = \langle -dn, dX \rangle \), then we obtain,

\[
[2.3] \quad II = \frac{L}{\sqrt{\varepsilon_1 \varepsilon_3}} du^2 + \left( \frac{1}{\sqrt{\varepsilon_1 \varepsilon_4}} + \frac{1}{\sqrt{\varepsilon_2 \varepsilon_3}} \right) M dudv + \frac{N}{\sqrt{\varepsilon_2 \varepsilon_4}} dv^2
\]

Equation [2.3] is called S.F.F. quadratic form of \( M \) at \( P \).

**Corollary 2.1.**

a) If \( X_u \) is time-like and \( X_v \) is space-like (resp. \( X_u \) space-like and \( X_v \) time-like) then surface is space-like. Thus F.F.F. for \( F=0 \),

\[
I = -Edu^2 + Gdv^2 \quad (\text{resp. } I = Edu^2 - Gdv^2)
\]

b) If surface is time-like then F.F.F.(resp. S.F.F.) is

\[
I = Edu^2 + 2Fduvdv + Gdv^2, \quad (\text{resp. } II = du^2 + Mdudv + dv^2)
\]

**3. WEINGARTEN AND OLÎN-RODRIGUES FORMULAS**

Let \( M \) be a surface which has been defined with vectorial function \( X = X(u, v) \) and \( n(u, v) \) be unit normal vector at \( P(u, v) \) on \( M \). Even though \( n_u \) and \( n_v \) perpendicular to \( n(u, v) \), these vectors are parallel to the tangent plane at \( P \) so we can write \( X_u \) and \( X_v \) vectors in linear combination of \( n_u \) and \( n_v \) as,

\[
n_u = a_{11}X_u + a_{12}X_v \\
n_v = a_{21}X_u + a_{22}X_v
\]

We can find coefficients multiplying these equations by \( X_u \) and \( X_v \) both side and simplifying we get,
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\[
a_{11} = \frac{1}{EG - F^2} \begin{pmatrix}
MF & LG \\
\sqrt{e_2 e_3} & \sqrt{e_1 e_3}
\end{pmatrix}, \quad a_{12} = \frac{1}{EG - F^2} \begin{pmatrix}
LF & ME \\
\sqrt{e_1 e_4} & \sqrt{e_2 e_4}
\end{pmatrix}
\]

\[
a_{21} = \frac{1}{EG - F^2} \begin{pmatrix}
NF & MG \\
\sqrt{e_2 e_4} & \sqrt{e_2 e_3}
\end{pmatrix}, \quad a_{22} = \frac{1}{EG - F^2} \begin{pmatrix}
MF & NE \\
\sqrt{e_2 e_4} & \sqrt{e_2 e_3}
\end{pmatrix}
\]

where \( a_{11}, a_{12}, a_{21}, a_{22} \) coefficients are called Weingarten coefficients and we show in matrix as \( S = [a_{ij}] \).

So the Gauss curvature of the surface will be follow.

\[
K = \frac{4LN}{\det I} \left( \frac{1}{\sqrt{e_1 e_2}} \frac{1}{\sqrt{e_2 e_3}} \right)^2 M^2
\]

If \( F = 0 \) and \( M = 0 \) than,

\[
K = \frac{\det II}{\det I} = \frac{LN}{\sqrt{e_2 e_4} EG}
\]

and mean curvature of \( M \) is

\[
H = \text{trc}(S) = \frac{1}{EG - F^2} \left( \frac{MF}{\sqrt{e_2 e_3}} - \frac{LG}{\sqrt{e_1 e_3}} + \frac{MF}{\sqrt{e_2 e_4}} - \frac{NE}{\sqrt{e_2 e_3}} \right)
\]

and since \( F = 0 \) and \( M = 0 \) than,

\[
H = \text{trc}(S) = -\frac{1}{EG} \left( \frac{MG}{\sqrt{e_2 e_3}} + \frac{NE}{\sqrt{e_2 e_3}} \right)
\]

**Corollary 3.1.**

\( a \) The matrix \( S \) of the time-like surfaces for both \( F \neq 0 \) and \( F = 0 \) will be as follows respectively.

\[
\begin{pmatrix}
MF - LG & LG - ME \\
NF - MG & MF - NE
\end{pmatrix}, \quad (\text{resp.} \quad \begin{pmatrix}
LG & ME \\
MG & NE
\end{pmatrix})
\]

\( b \) Space-like surface’s shape operator matrix for \( F \neq 0 \) and \( F = 0 \) has complex coefficients.

If coordinate lines are perpendicular at every point on the surface then \( \langle X_u, X_v \rangle = 0 \) and \( F = 0 \). So the new equations are;

\[
n_u = \frac{-L}{\sqrt{e_1 e_3}} X_u + \frac{-M}{\sqrt{e_2 e_4}} G, \\
n_v = \frac{-M}{\sqrt{e_2 e_3}} X_u + \frac{-N}{\sqrt{e_2 e_3}} G
\]

If \( F = 0 \) and \( M = 0 \) then we get

\[3.1\]

\[
n_u + \frac{r}{\sqrt{e_1 e_3}} X_u = 0, \quad n_v + \frac{r}{\sqrt{e_2 e_3}} X_v = 0
\]

where \( r \) and \( \bar{r} \) are

\[
r = \frac{L}{E}, \quad \bar{r} = \frac{N}{G}
\]

\[3.1\] equations which we have obtained are called Olin-Rodrigues formulae.

**Corollary 3.2.** Olin-Rodrigues equations of the time-like and space-like surface are

\[
n_u + r X_u = 0, \quad n_v + \bar{r} X_v = 0
\]
References


