3 BOYUTLU LORENTZ UZAYINDA KÜRESEL ALANLAR İÇİN GEOMETRİK BİR İNVARYANT

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ÖZET
Bu çalışmada, 3-boyutlu Lorentz uzayında 1-parametreli Lorentz küresel hareketi tanımlanarak, bu hareket sırasında space-like birim Lorentz küresi üzerinde elde edilen küresel alanlar üzerindeki bağıntı verilmiştir ve bu alanlar yardımcıla geometrik bir invaryant elde edilmiştir.

Anahtar Kelimeler: Lorentz küresi, Lorentz küresel alanı, Kapalı Lorentz küresel hareketi, 1-parametreli kapalı küresel hareket.

A GEOMETRICAL INVARIANT FOR SPHERICAL AREAS IN LORENTZ 3-SPACE $L^3$

ABSTRACT
In this study, by defining the one-parameter closed spherical Lorentz motion in 3-dimensional Lorentz space, we give the relation between spherical areas, generated by this motion on space-like unit Lorentzian sphere.

Key Words: Lorentz sphere, Lorentz spherical area, Motion of closed Lorentz sphere, One Parameter closed spherical motion.

1. INTRODUCTION

A 3–dimensional vector space $L = L^3_1$ with scalar product $\langle \cdot, \cdot \rangle_L$ of index 1 is called a Lorentzian vector space. A vector $X$ of $L^3_1$ is said to be space-like if $\langle X, X \rangle_L > 0$, time-like if $\langle X, X \rangle_L < 0$ and light-like or null if $\langle X, X \rangle_L = 0$. A curve in $L^3_1$ is called space-like (time-like or null, respectively) if its tangent vector is space-like (time-like or null, respectively).

Let $X = (X_i)$ and $Y = (Y_i)$ be the vectors in a 3–dimensional Lorentz vector space $L^3_1$, then the scalar product of $X$ and $Y$ is defined by

$$\langle X, X \rangle_L = X_1Y_1 + X_2Y_2 - X_3Y_3,$$

which is called a Lorentzian product. Furthermore, a Lorentzian cross product $X \Lambda_L Y$ is given by

$$X \Lambda_L Y = (-X_2Y_3 + X_3Y_2, X_3Y_1 - X_1Y_3, X_1Y_2 - X_2Y_1).$$

For $X \in L^3_1$, the norm of $X$ is defined by $\|X\|_L = \sqrt{\langle X, X \rangle_L}$, and $X$ is called a unit vector if $\|X\|_L = 1$.
Lorentzian motion $B' = K/K'$ of the moving unit Lorentz sphere $K$ with the fixed center $O$ with respect to the fixed unit Lorentz sphere of the same center defines a direct unique Lorentzian motion about the fixed point $O$. Hence, Lorentzian spherical motion is a spatial motion in Lorentz $3 - \text{space}$ . During the Lorentz motion $B' = K/K'$, which leaves the center $O$ fixed, the orthonormal, positive directed two coordinate system

\[ \{O; \vec{E}_1, \vec{E}_2, \vec{E}_3\}, \{O; \vec{E}_1', \vec{E}_2', \vec{E}_3'\} \]

represent respectively moving $K$ and fixed $K'$ Lorentzian sphere. These two coordinate systems depend invariantly to unit Lorentz spheres $K$ and $K'$. Let’s denote the matrices

\[
E = \begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 \end{bmatrix}, \quad E' = \begin{bmatrix} \vec{E}_1' \\ \vec{E}_2' \\ \vec{E}_3' \end{bmatrix}
\]

Since these two systems are orthonormal, for $A$ being an orthogonal Lorentzian matrix, we have

\[ E = AE', \]

where

\[ A^{-1} = \varepsilon A^T \varepsilon \]

and

\[
\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]

is the sign matrix in Lorentzian $3 - \text{space}$. Let the matrices $E, E'$ and $A$ be differentiable functions in sufficient order of a parameter $t \in \mathbb{R}$. Here $A$ is not periodic, but during the Lorentzian spherical motion, the spherical curve $(X)$ drawn on $K'$ by the point $X$ taken from the moving sphere $K$ is periodic. Thus the motion $B' = K/K'$ of moving Lorentzian motion $K$ with respect to fixed Lorentz sphere $K'$ is called as one parameter closed spherical motion.

In this study, $\vec{E}_1, \vec{E}_2, \vec{E}_3$ and $\vec{E}_1', \vec{E}_2', \vec{E}_3'$ are taken to be space-like vektors and $\vec{E}_3, \vec{E}_3'$ are taken to be timelike vektors. The directional Lorentzian 2-manifold, we consider is the surface of space like Lorentzian sphere

\[ x^2 + y^2 - z^2 = r^2, \quad r = \text{const}. \]

Which has the time-like vector as normal. The curves on the surface are uniform closed spacelike curves. Computatin are made for the upper half of the space-like Lorentzian sphere. The area of upper half part of unit Lorentzian sphere K is $1/1$ [1].

Let $X$ be an arbitrary point on $K$ moving Lorentz sphere. Then, the point $X$ draw the closed $(X)$ Lorentz curve during the one parameter closed Lorentz spherical motion $B' = K/K'$. The spherical area bounded by the closed Lorentzian curve $(X)$ is

\[ \int_{\mathcal{X}} = 2\pi + A_{\mathcal{X}} \]

[2,3].
2. A GEOMETRICAL INVARIANT FOR SPHERICAL AREAS IN LORENTZIAN 3-SPACE $L^3$

**Theorem 2.1.** Two constant points $M$ and $N$ on moving unit Lorentzian sphere $K$ usually draw two closed curves on constant unit Lorentz sphere $K'$ during the closed Lorentz motion with one parameter $B' = K/K'$. Let these curves be $(M)$ and $(N)$, and the Lorentzian spherical areas bounded by these two curves be $F_M$ and $F_N$.

Consider another point $X$ of $K$ on $MN$ arc with constant length of great Lorentzian circle of Lorentzian sphere $K$. During the same motion, $X$ also draws another closed curve $(X)$ on constant Lorentzian sphere $K'$. Let $F_X$ be the area bounded by the closed curve $(X)$. In this, the relation between these areas are;

$$F_X = \frac{1}{2} \left( F_M + F_N + \wedge_{MX+NX} \right)$$

**Proof.** From the formula in [3] we have,

$$F_M = 2\pi + \wedge_M$$
$$F_N = 2\pi + \wedge_N$$
$$F_X = 2\pi + \wedge_X$$

![Figure 2.1](image-url) The area $MN$, taken on great Lorentzian circle on Lorentzian sphere ($z > 0$).
\( \vec{M}, \vec{N} \) and \( \vec{X} \) are the position vectors of \( M, N \) and \( X \) respectively, and from Figure 2.1, We have

\[
\vec{N} = \vec{M} + \vec{MN}, \quad \vec{X} = \vec{M} + \vec{MX}, \quad \vec{X} = \vec{N} + \vec{NX}
\]  

[2.2]

From [2.1] and [2.2], we may write that

\[
F_N = F_M + \wedge_{MN},
\]

\[
F_X = F_M + \wedge_{MX}
\]  

[2.3]

and

\[
F_X = F_N + \wedge_{NX}
\]  

[2.4]

From the last two terms of \( F_X \) we

\[
F_X = \frac{1}{2} \left\{ F_M + F_N + \wedge_{MN} + \wedge_{NX} \right\}
\]

This equation is equivalent to Holditch theorem. That is, the relation between the closed spherical Lorentzian areas bounded by the closed Lorentzian curves \( M, N \) and \( X \) are independent from the closed Lorentzian motion \( B' = K/K' \).

An important result of Holditch theorem is the special case of \( F_M = F_N \). The end points \( M \) and \( N \) either draw the curves having equal area or the same curve \( \gamma \) on \( K' \) Lorentz sphere. So from [2.3], we have

\[
\wedge_{MN} = 0
\]  

[2.5]

is obtained. Here \( \vec{S} \) is a Steiner vector.

**Corollary 2.1.** During the closed Lorentzian spherical motion with one-parameter, \( \wedge_{MN} = 0 \) iff the end points \( M \) and \( N \) draw the same spherical curve.

**Corollary 2.2.** Let’s consider the areas bounded by some different points not on the same great circle of moving Lorentzian sphere \( K \). These areas are equal iff the points belong to same curve.

**REFERENCES**

