Dumlupınar Üniversitesi Fen Bilimleri Enstitüsü Dergisi
Sayı:11 Eylül 2006 ISSN - 1302 - 3055

E_1^3'de TIME-LIKE BİR EĞRİ İÇİN ÖZEL BİR FRENET HAREKETİNİN DARBOUX VEKTÖRLERİ ÜZERİNE

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ÖZET

Bir parametreli özel Frenet hareketi, Bottema tarafından 3-boyutlu Öklid uzayı E^3 de verilmiştir [1]. Bu çalışmada, E^3 deki [1] Frenet hareketinin, E_1^3, 3-boyutlu Minkowski uzayına bir genelleştirilmesi veriliyor. İlk olarak, E^3 deki bu hareket, time-like bir eğri için E_1^3 uzayında tanımlandı daha sonra, bu hareketin Darboux vektörleri [2] sabit ve hareketli uzaylar için hesaplandı.

Anahtar Kelimeler: *Causal karakter, Darboux vektörleri, Özel Frenet hareketi, Time-like curve*

ON DARBOUX VECTORS OF A SPECIAL FRENET MOTION FOR A TIME-LIKE CURVE IN E_1^3

ABSTRACT

A special Frenet motion with an one-parameter in E^3 has been given by Bottema [1]. In this study, we give a generalization of [1] to E_1^3, Minkowski 3-space. Firstly, this motion in E^3 is defined in E_1^3 for a time-like space curve and then Darboux vectors [2] of this motion is calculated for fixed and moving spaces in E_1^3 .

Keywords: *Causal character, Darboux vector, Frenet motion, Lorentz space, Time-like curve.*

1. INTRODUCTION

An one-parameter motion of a body in Minkowski 3-space is generated by the transformation

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ 1 \end{bmatrix} \tag{1.1}$$

where A is a semi-orthogonal matrix , $A \in SO(3,1)$, and c is the displacement vector of the origin. A ve c are C^{∞} functions of a real parameter s. x_0, X, c are $n \times 1$ matrices and

$$SO(3,1) = \left\{ A \in R_3^3 \middle| A^{-1} = \varepsilon A^T \varepsilon, \det A = 1, \varepsilon = diag(-1,1,1) \right\} \quad [3].$$

x_0 and X respectively correspond to the position vectors of the same point P, with respect to the orthogonal coordinate systems of the moving space H and the fixed space H'. At the unital time $s = s_0$ we consider the coordinate systems of H and H' are coincident.

We define a special motion in terms of a timelike space curve α given in the fixed space H'. This motion is such that the moving frame O_{xyz} moves with O along α while rotating so that the x and y axes always coincide with, respectively, the tangent and principal normal of α. This means that as O coincides with a point Q of α, the O_{xyz} frame coincides with the Frenet trihedron at Q : $Q_{\xi\eta\rho}$. This trihedron consists of the tangent Q_{ξ}, the principal normal Q_{η}, and the binormal Q_{ρ}, which are three mutually orthogonal axes. Obviously, the geometry of this motion is completely defined by α .

For our special motion, because O moves along α , X=$\boldsymbol{c}$ represents the time-like space curve α . We will use the arc length s of α as the motion parameter, and use primes to denote derivatives with respect to s.

The Lorentzian or Minkowski 3-space E_1^3 is the Euclidean 3-space provided with the Lorentzian inner product

$$< x, y >:= -x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. An arbitrary vector $x = (x_1, x_2, x_3)$ in E_1^3 may have one of three Lorentzian causal characters:

if $< x, x > > 0$ or $\vec{x} = 0$, it is space-like,
if $< x, x > < 0$ it is time-like and
if $< x, x > = 0$ it is null (light-like) [3].

Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 is locally space-like, time-like or null, for each $s \in I \subset R$. Recall that the pseudo-norm of an arbitrary vector $\vec{x} \in E_1^3$ is given by $\|x\| = \sqrt{|< x, x >|}$, and that the velocity v of the curve α is $v = \|\alpha'(s)\|$. Therefore, α is a unit speed curve if and only if $< \alpha'(s), \alpha'(s) >= \pm 1$.

Denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha = \alpha(s)$ parameterized by a pseudo-arclength parameter s, i.e. $< \alpha'(s), \alpha'(s) >= \pm 1$. If

$T=(t_1,t_2,t_3)$, $N=(n_1,n_2,n_3)$ and $B=(b_1,b_2,b_3)$ are the unit vectors along Q_ξ, Q_η and Q_ρ, Lorentzian differential geometry gives us

$$T(s)=\alpha'(s)\,,\,N(s)=\alpha''(s)/\|\alpha''\|\,,\,B(s)=T(s)\wedge N(s)\,, \tag{1.2}$$

where $k=\|\alpha''\|$ is the curvature of curve α and "$\wedge$" denotes vector product in E^3. If $\alpha=\alpha(s)$ is a time-like curve, i.e., if *T* is a time-like vector, then the Frenet formulae read

$$T'=kN,\quad N'=kT+\tau B,\quad B'=-\tau N \tag{1.3}$$

$<T,T>=-1,\quad <N,N>=<B,B>=1,\quad <T,N>=<T,B>=<N,B>=0$,

here, *k* the curvature and τ the torsion of the timelike curve α [4]. Then we have

$$A=\begin{bmatrix} t_1 & n_1 & b_1 \\ t_2 & n_2 & b_2 \\ t_3 & n_3 & b_3 \end{bmatrix}. \tag{1.4}$$

It can be defined that an one-parameter *special* motion of a body in Lorentzian space H' is generated by the transformation

$$X=Ax_0+c \tag{1.5}$$

where $A\in SO(3,1)$ and x_0, X, c are *3x1* real matrices [1].

2. DARBOUX VECTORS (MATRICES)

Theorem: Let the motion H/H' be represented by the equation (1.4). Then the components of Darboux vectors of the motion H/H', respectively, are

$$\omega=(kb_1-\tau\, t_1,\; kb_2-\tau\, t_2,\; kb_3-\tau\, t_3)$$

and

$$w=(-\tau, 0, k)\,.$$

Proof: From (1.1) we obtain

$$x_0=A^{-1}(X-c) \tag{2.1}$$

and as $A^{-1}=\varepsilon A^{T}\varepsilon$ and det A = + 1, we have from (1.4) ve (2.1), eliminating x_0

$$X' = \Omega(X-c)+c' \tag{2.2}$$

with $\Omega = A'A^{-1}$, or explicitly, by means of (1.2),

$$\Omega=\begin{bmatrix} t_1' & n_1' & b_1' \\ t_2' & n_2' & b_2' \\ t_3' & n_3' & b_3' \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t_1 & t_2 & t_3 \\ n_1 & n_2 & n_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Omega=\begin{bmatrix} t_1' & n_1' & b_1' \\ t_2' & n_2' & b_2' \\ t_3' & n_3' & b_3' \end{bmatrix}\begin{bmatrix} t_1 & -t_2 & -t_3 \\ -n_1 & n_2 & n_3 \\ -b_1 & b_2 & b_3 \end{bmatrix}$$

$$\Omega=\begin{bmatrix} t_1't_1-n_1'n_1-b_1'b_1 & -t_1't_2+n_1'n_2+b_1'b_2 & -t_1't_3+n_1'n_3+b_1'b_3 \\ t_2't_1-n_2'n_1-b_2'b_1 & -t_2't_2+n_2'n_2+b_2'b_2 & -t_2't_3+n_2'n_3+b_2'b_3 \\ t_3't_1-n_3'n_1-b_3'b_1 & -t_3't_2+n_3'n_2+b_3'b_2 & -t_3't_3+n_3'n_3+b_3'b_3 \end{bmatrix}.$$

One can find as follows

$$T = N\times B = (t_1,t_2,t_3) = n_3b_2-n_2b_3,\; n_3b_1-n_1b_3,\; n_1b_2-n_2b_1 \tag{2.3}$$

$$B = T\times N = (b_1,b_2,b_3) = (n_2t_3-n_3t_2,\; n_1t_3-n_3t_1,\; n_2t_1-n_1t_2) \tag{2.4}$$

$$\left.\begin{array}{l} T' = (t_1',t_2',t_3') = (kn_1,kn_2,kn_3) \\ N' = (n_1',n_2',n_3') = (kt_1+\tau b_1, kt_2+\tau b_2, kt_3+\tau b_3) \\ B' = (b_1',b_2',b_3') = (-\tau\ n_1, -\tau\ n_2, -\tau\ n_3). \end{array}\right\}. \tag{2.5}$$

By using (2.3), (2.4) and (2.5), we get

$$\Omega=\begin{bmatrix} t_1't_1-n_1'n_1-b_1'b_1 & -t_1't_2+n_1'n_2+b_1'b_2 & -t_1't_3+n_1'n_3+b_1'b_3 \\ t_2't_1-n_2'n_1-b_2'b_1 & -t_2't_2+n_2'n_2+b_2'b_2 & -t_2't_3+n_2'n_3+b_2'b_3 \\ t_3't_1-n_3'n_1-b_3'b_1 & -t_3't_2+n_3'n_2+b_3'b_2 & -t_3't_3+n_3'n_3+b_3'b_3 \end{bmatrix}$$

$$\Omega=\begin{bmatrix} (kn_1)t_1-(kt_1+\tau b_1)n_1 & -(kn_1)t_2+(kt_1+\tau b_1)n_2 & -(kn_1)t_3+(kt_1+\tau b_1)n_3 \\ -(-\tau n_1)b_1 & +(-\tau n_1)b_2 & +(-\tau n_1)b_3 \\ (kn_2)t_1-(kt_2+\tau b_2)n_1 & -(kn_2)t_2+(kt_2+\tau b_2)n_2 & -(kn_2)t_3+(kt_2+\tau b_2)n_3 \\ -(-\tau n_2)b_1 & +(-\tau n_2)b_2 & +(-\tau n_2)b_3 \\ (kn_3)t_1-(kt_3+\tau b_3)n_1 & -(kn_3)t_2+(kt_3+\tau b_3)n_2 & -(kn_3)t_3+(kt_3+\tau b_3)n_3 \\ -(-\tau n_3)b_1 & +(-\tau n_3)b_2 & +(-\tau n_3)b_3 \end{bmatrix}$$

$$\Omega=\begin{bmatrix} +kn_1t_1-kt_1n_1 & -kn_1t_2+kt_1n_2 & -kn_1t_3+kt_1n_3 \\ -\tau b_1n_1+\tau n_1b_1 & +\tau b_1n_2-\tau n_1b_2 & +\tau b_1n_3-\tau n_1b_3 \\ +kn_2t_1-kt_2n_1 & -kn_2t_2+kt_2n_2 & -kn_2t_3+kt_2n_3 \\ -\tau b_2n_1+\tau n_2b_1 & +\tau b_2n_2-\tau n_2b_2 & +\tau b_2n_3-\tau n_2b_3 \\ +kn_3t_1-kt_3n_1 & -kn_3t_2+kt_3n_2 & -kn_3t_3+kt_3n_3 \\ -\tau b_3n_1+\tau n_3b_1 & +\tau b_3n_2-\tau n_3b_2 & +\tau b_3n_3-\tau n_3b_3 \end{bmatrix}$$

$$\Omega=\begin{bmatrix} 0 & +k(t_1n_2-n_1t_2) & -k(n_1t_3-t_1n_3) \\ & -\tau(n_1b_2-b_1n_2) & +\tau(n_1b_3-b_1n_3) \\ +k(n_2t_1-t_2n_1) & 0 & -k(n_2t_3-t_2n_3) \\ -\tau(b_2n_1-n_2b_1) & & +\tau(b_2n_3-n_2b_3) \\ +k(n_3t_1-t_3n_1) & +k(t_3n_2-n_3t_2) & 0 \\ -\tau(n_3b_1-b_3n_1) & -\tau(n_3b_2-b_3n_2) & \end{bmatrix}$$

$$\Omega=\begin{bmatrix} 0 & (kb_3-\tau t_3) & -(kb_2-\tau t_2) \\ (kb_3-\tau t_3) & 0 & -(kb_1-\tau t_1) \\ -(kb_2-\tau t_2) & (kb_1-\tau t_1) & 0 \end{bmatrix} \leftrightarrow \omega=(kb_1-\tau t_1\,,kb_2-\tau t_2\,,kb_3-\tau t_3). \quad (2.6)$$

Ω is a semi skew–symmetric matrix as $\Omega = -\varepsilon\,\Omega^T\varepsilon$. Its components with respect to the moving frame follow from $w = A^{-1}\,\omega$ [5], and we obtain the vector

$$
\begin{aligned}
w &= \varepsilon\,A^T\varepsilon\,\omega \\
&= \begin{bmatrix} t_1 & -t_2 & -t_3 \\ -n_1 & n_2 & n_3 \\ -b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} k\,b_1 - \tau\,t_1 \\ k\,b_2 - \tau\,t_2 \\ k\,b_3 - \tau\,t_3 \end{bmatrix} \\
&= \begin{bmatrix} \tau(-t_1^2 + t_2^2 + t_3^2) - k(-t_1b_1 + t_2b_2 + t_3b_3) \\ k(-n_1b_1 + n_2b_2 + n_3b_3) - \tau(-n_1t_1 + n_2t_2 + n_3t_3) \\ \tau(-b_1^2 + b_2^2 + b_3^2) - k(-t_1b_1 + t_2b_2 + t_3b_3) \end{bmatrix} \\
&= \begin{bmatrix} \tau < T,T > -k < T,B > \\ k < N,B > -\tau < T,N > \\ \tau < B,B > -k < T,B > \end{bmatrix} \\
&= \begin{bmatrix} \tau(-1) & 0 & k \end{bmatrix}^T
\end{aligned}
$$

and so

$$w = (-\tau, 0, k)\,.$$

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