ON WEAK SYMMETRIES OF \((k, \mu)\)-CONTACT METRIC MANIFOLDS

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ABSTRACT
In this study, we consider weakly symmetric and weakly Ricci-symmetric \((k, \mu)\)-contact metric manifolds. We find necessary conditions in order that a \((k, \mu)\)-contact metric manifold be weakly symmetric and weakly Ricci symmetric.


Key Words: Weakly symmetric , weakly Ricci-symmetric, \((k, \mu)\)-contact metric manifolds.

\( (k, \mu)\)-DEĞME METRİK MANİFOLDLARIN ZAYIF SİMETRİLERİ ÜZERİNE

ÖZET
Bu çalışmada, zayıf simetrik ve zayıf Ricci-simetrik \((k, \mu)\)-değme metrik manifoldlar göz önüne aldık. \((k, \mu)\)-değme metrik manifoldların zayıf simetrik ve zayıf Ricci-simetrik olması için gerekli şartları bulduk.

Anahtar Kelimeler: Zayıf simetrik, zayıf Ricci-simetrik, \((k, \mu)\)-değme metrik manifoldlar.

1. INTRODUCTION
Let \((M, g)\) be an \(n\)-dimensional, \(n \geq 2\), semi-Riemannian manifold of class \(C^\infty\). We denote by \(\nabla\) the Levi-Civita connection. Then we have

\[ R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z. \]

The Riemannian-Christoffel tensor and the Ricci tensor of \((M, g)\) are defined by

\[ R(X,Y,Z,W) = g(R(X,Y)Z,W) \]

and

\[ S(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i) \]

respectively, where \(X, Y, Z, W \in \chi(M)\), where \(\chi(M)\) is the Lie algebra of vector fields on \(M\) and \(\{e_1, e_2, ..., e_n\}\) is a local orthonormal basis for the vector fields on \(M\).

A non-flat differentiable manifold \((M^n, g)\), \((n > 3)\), is called pseudosymmetric if there exists a 1-form \(\alpha\) on \(M\) such that
(\nabla_X R)(Y, Z, W) = 2\alpha(X)R(Y, Z)W + \alpha(Y)R(X, Z)W + \alpha(Z)R(Y, X)W + \alpha(W)R(Y, Z)X + g(R(Y, Z)W, X)A,

where \(X, Y, Z, W \in \chi(M)\) are arbitrary vector fields and \(A \in \chi(M)\) is the vector field corresponding through \(g\) to the 1-form \(\alpha\) which is given by \(g(X, A) = \alpha(A)\).

A non-flat differentiable manifold \((M^n, g)\), \((n > 3)\), is called weakly symmetric if there exists a vector field \(P\) and 1-forms \(\alpha, \beta, \gamma, \delta\) on \(M\) such that

\[(\nabla_X R)(Y, Z, W) = \alpha(X)R(Y, Z)W + \beta(Y)R(X, Z)W + \gamma(Z)R(Y, X)W + \delta(W)R(Y, Z)X + g(R(Y, Z)W, X)P,\]

holds for all vector fields \(X, Y, Z, W \in \chi(M)\) ([10] and [11]). A weakly symmetric manifold \((M, g)\) is pseudosymmetric if \(\beta = \gamma = \delta = 0\) and \(P = A\), locally symmetric if \(\alpha = \beta = \gamma = \delta = 0\) and \(P = 0\). A weakly symmetric is said to be proper if at least one of the 1-forms \(\alpha, \beta, \gamma, \delta\) is not zero or \(P \neq 0\).

A differentiable manifold \((M^n, g)\), \((n > 3)\), is called weakly Ricci-symmetric if there exists 1-forms \(\varepsilon, \sigma, \rho\) such that the condition

\[(\nabla_X S)(Y, Z) = \varepsilon(X)S(Y, Z) + \sigma(Y)S(X, Z) + \rho(Z)S(X, Y),\]

holds for all vector fields \(X, Y, Z \in \chi(M)\) ([10] and [11]). If \(\varepsilon = \sigma = \rho\) then \(M\) is called pseudo Ricci-symmetric ([5]).

From (2), an easy calculation shows that if \(M\) is weakly symmetric then we have

\[(\nabla_X S)(Z, W) = \alpha(X)S(Z, W) + \beta(R(X, Z)W) + \varepsilon(Z)S(X, W) + \sigma(W)S(X, Z) + p(R(X, W)Z),\]

where \(P\) is defined by \(p(X) = g(X, P)\) for all \(X \in \chi(M)\) ([11]).

In [11], the authors considered weakly symmetric and weakly Ricci-symmetric Einstein and Sasakian manifolds. In [5], the authors studied weakly symmetric and weakly Ricci-symmetric K-contact manifolds. Also, in [1], the authors studied pseudosymmetric contact metric manifolds of Chaki type. In this study we consider weakly symmetric and weakly Ricci-symmetric \((k, \mu)\)-contact metric manifolds.

2. PRELIMINARIES

Let \(M\) be a \((2n+1)\)-dimensional contact metric manifold with structure tensors \((\varphi, \zeta, \eta, g)\). Then the structure tensors satisfy the following equations

\[
\begin{align*}
\varphi^2 &= -I + \eta \otimes \zeta, \\
\eta(\zeta) &= 1, \\
\varphi \zeta &= 0, \\
g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), \\
g(\varphi X, Y) &= d\eta(X, Y),
\end{align*}
\]

for any vector fields \(X, Y\) on \(M\) [2]. The \((1,1)\)-tensor field \(h\) defined by \(h = -\frac{1}{2}L_\varphi\), where \(L\) denotes Lie differentiation. Then the vector field \(\zeta\) is Killing if and only if \(h\) vanishes. It is well known that \(h\) and \(\varphi h\) are symmetric operators, \(h\) anti-commutes with \(\varphi\) (i.e., \(\varphi h + h\varphi = 0\)), \(h\zeta = 0\), \(\eta h = 0\), \(tr h = 0\) and \(tr \varphi h = 0\), where \(tr h\) denotes the trace of \(h\). Since \(h\) anti-commutes with \(\varphi\), if \(X\) is an eigenvector of \(h\) corresponding to the eigenvalue
then \( \varphi X \) is also an eigenvector of \( h \) corresponding to the eigenvalue \( -\lambda \). Moreover, for any contact metric manifold \( M \), the following is satisfied

\[
\nabla X \xi = -\varphi X \varphi h X
\]

(7)

here \( \nabla \) is the Riemannian connection of \( g \). If \( \xi \) is Killing on a contact metric manifold \( M \), then \( M \) is said to be a K-contact Riemannian manifold. We also recall that on a K-contact Riemannian manifold it is valid \( R(X, \xi) \xi = -\eta(X) \xi \).

The \((k, \mu)\)-nullity distribution of a Riemannian manifold \((M, g)\) for a real numbers \( k, \mu \) is a distribution

\[
N(k, \mu) : p \rightarrow N_p(k, \mu) = \{ Z \in T_p(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY] \}
\]

for any \( X, Y \in T_p(M) \). We consider that \( M \) is a contact metric manifold with belonging \( \xi \) to the \((k, \mu)\)-nullity distribution i.e. [3],

\[
R(X, Y) \xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],
\]

(8)

\[
R(\xi, X) Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX],
\]

(9)

\[
S(X, \xi) = 2nk\eta(X),
\]

(10)

\[
Q^{\xi} = 2nk\xi.
\]

(11)

In particular, on a contact metric manifold, \( M \) is Sasakian if and only if \( k=1 \) and \( \mu=0 \).

3. MAIN RESULTS

In this chapter we investigate weakly symmetric and weakly Ricci-symmetric \((k, \mu)\)-contact metric manifolds. Firstly we have:

**Theorem 1** There exists no weakly symmetric \((k, \mu)\)-contact metric manifold \( M^{2n+1} \), \((k \neq 0)\), \( n>1 \), if \( \alpha + \gamma + \delta \) is not everywhere zero.

**Proof.** Assume that \( M^{2n+1} \) is a weakly symmetric \((k, \mu)\)-contact metric manifold. Putting \( W=\xi \) in (4) we get

\[
(\nabla X)(Z, \xi) = k\alpha(X)S(Z, \xi) + \beta(R(X, Z) \xi)
\]

(12)

\[
+ \gamma(Z)S(X, \xi) + \delta(\xi)S(X, Z) + p(R(X, \xi) Z) + p(R(X, \xi) \xi).
\]

So using (8), (9) and (10) we have

\[
(\nabla X)(Z, \xi) = 2nk\alpha(X)\eta(Z) + k\beta(X)\eta(Z) + k\beta(Z)\eta(X)
\]

(13)

\[
+ \mu\beta(hX)\eta(Z) + \mu\beta(hZ)\eta(X) + 2nk\gamma(Z)\eta(X)
\]

\[
+ \delta(\xi)S(X, Z) + k\eta(Z)p(X) - kg(X, Z)p(\xi)
\]

\[
+ \mu\eta(Z)p(hX).
\]
By the covariant differentiation of the Ricci tensor $S$, the left side can be written as

$$\nabla_{\nabla XS}(Z, \xi) = \nabla XS(Z, \xi) - S(\nabla XZ, \xi) - S(Z, \nabla X\xi).$$

By the use of (7), (10) and the parallelity of the metric tensor $g$ we have

$$\nabla XS(Z, \xi) = -2nk\varphi(X, Z) - 2nk\varphi(hX, Z) + S(\varphi X, Z) + S(\varphi hX, Z).$$

Comparing the right hand sides of (13) and (14), we obtain

$$-2nk\varphi(X, Z) - 2nk\varphi(hX, Z) + S(\varphi X, Z) + S(\varphi hX, Z) = 2nk\alpha(X)\eta(W) + k\beta(X) - k\beta(\xi)\eta(X) + \mu\beta(hX) + 2nk\gamma(\xi)\eta(W) + 2nk\delta(W) + k\eta(pX) - k\eta(\xi)p(hX).$$

Putting $X=Z=\xi$ in (15) and using (5), (6) and (10) we get

$$2nk[\alpha(\xi) + \gamma(\xi) + \delta(\xi)] = 0.$$  

Since $n>1$ and $k\neq 0$, we obtain

$$\alpha(\xi) + \gamma(\xi) + \delta(\xi) = 0.$$  

So vanishing of the 1-form $\alpha+\gamma+\delta$ over the vector field $\xi$ necessary in order that $M$ be a $(k, \mu)$-contact metric manifold.

Now we will show that $\alpha+\gamma+\delta=0$ holds for all vector fields on $M$.

In (4), taking $Z=\xi$, similar to the previous calculations it follows that

$$-2nk\varphi(X, W) - 2nk\varphi(hX, Z) + S(\varphi X, W) + S(\varphi hX, Z)$$

$$= 2nk\alpha(X)\eta(W) + k\beta(X) - k\beta(\xi)\eta(X) + \mu\beta(hX) + 2nk\gamma(\xi)\eta(W) + 2nk\delta(W) + k\eta(pX) - k\eta(\xi)p(hX).$$

Replacing $W$ with $\xi$ in (17) and by making use of (5), (8) and (10) we have

$$2nk\alpha(X) + k\beta(X) - k\beta(\xi)\eta(X) + \mu\beta(hX) + 2nk\gamma(\xi)\eta(W) + 2nk\delta(W) + k\eta(pX) - k\eta(\xi)p(hX) = 0.$$  

Putting $X=\xi$ in (17) and by virtue of (5), (8) and (10) we find

$$2nk\alpha(\xi)\eta(W) + 2nk\gamma(\xi)\eta(W) + 2nk\delta(W)$$

$$= 2nk\alpha(\xi)\eta(W) + k\beta(\xi) - k\beta(\xi)\eta(X) + \mu\beta(hX) + 2nk\gamma(\xi)\eta(W) + 2nk\delta(W) + k\eta(pX) - k\eta(\xi)p(hX).$$

$$= 0.$$  

$$2nk\alpha(\xi)\eta(W) + 2nk\gamma(\xi)\eta(W) + 2nk\delta(W) = 0.$$  

(19)
Replacing $W$ with $X$ in (19) and taking the summation with (18), in view of (16), we obtain

$$2nk\alpha(X)+k\beta(X)-k\beta(\xi)\eta(X)+\mu\beta(hX)+2nk\delta(Z)+2nk\gamma(\xi)\eta(Z) = 0.$$  

Now putting $X=\zeta$ in (15) we have

$$2nk(\xi)\eta(Z)+k(\xi)\eta(Z)-k\beta(Z)\eta(Z)-\mu\beta(hZ)+2nk\gamma(Z)+2nk\delta(\xi)\eta(Z) = 0.$$  

So replacing $Z$ with $X$ in (21) and taking the summation with (20), in view of (16), we find

$$2nk[\alpha(X)+\gamma(X)+\delta(X)] = 0.$$  

Since $n>1$ and $k\neq 0$, we get

$$\alpha(X)+\gamma(X)+\delta(X) = 0,$$

for all $X$. This implies $\alpha+\gamma+\delta=0$, which completes the proof of the theorem.

**Theorem 2** There exists no weakly Ricci-symmetric $(k, \mu)$-contact metric manifold $M^{2n+1}$, $(k\neq 0)$, $n>1$, if $\varepsilon+\sigma+\rho$ is not everywhere zero.

**Proof.** Assume that $M^{2n+1}$ is a weakly Ricci-symmetric $(k, \mu)$-contact metric manifold. Replacing $Z$ with $\xi$ in (3) and using (10) we have

$$\nabla X S(Y, \xi) = 2nk\varepsilon(X)\eta(Y)+2nk\sigma(Y)\eta(X)+\rho(\xi)S(X,Y).$$  

Replacing $Z$ with $Y$ in (14) and comparing the right hand sides of the equations (22) and (14) we obtain

$$-2nk\varepsilon(\phi X, Y)-2nk\sigma(\phi X, Z)+S(\phi X, Y)+S(\phi X, Z)$$  

$$= 2nk\varepsilon(X)\eta(Y)+2nk\sigma(Y)\eta(X)+\rho(\xi)S(X,Y).$$  

Taking $X=Y=\zeta$ in (23) and by making use of (5), (6) and (10) we get

$$2nk[\varepsilon(\zeta)+\sigma(\zeta)+\rho(\zeta)] = 0.$$  

which gives, (since $n>1$ and $k\neq 0$),

$$\varepsilon(\zeta)+\sigma(\zeta)+\rho(\zeta) = 0.$$  

Putting $X=\zeta$ in (23) we have

$$2nk\varepsilon(Y)[\varepsilon(\zeta)+\rho(\zeta)+2nk\sigma(Y) = 0.$$  

So by virtue of (24) this yields $2nk[\varepsilon(Y)\sigma(\zeta)+\sigma(Y)] = 0$, which gives us (since $n>1$ and $k\neq 0$)

$$\sigma(Y) = \sigma(\zeta)\eta(Y).$$

Similarly taking $Y=\zeta$ in (23) we also have
Applying (24) into the last equation we get

$$\varepsilon(X) = \varepsilon(\xi)\eta(X). \quad (26)$$

Since \((\nabla_{\xi} S)(\xi, X)=0\), then from (3) we obtain

$$2nk\eta(X)[\varepsilon(\xi)+\sigma(\xi)]+2nk\rho(X) = 0. \quad (27)$$

So by making use of (24), the equation (27) reduces to

$$\rho(X) = \rho(\xi)\eta(X). \quad (28)$$

Therefore the summation of the equations (25), (26) and (28) give us

$$\varepsilon(X) + \sigma(X) + \rho(X) = (\varepsilon(\xi) + \sigma(\xi) + \rho(\xi))\eta(X).$$

and then, from (24), it follows that

$$\varepsilon(X) + \sigma(X) + \rho(X) = 0,$$

for all \(X\). Thus \(\varepsilon + \sigma + \rho = 0\). Our theorem is proved.

REFERENCES


