SOME NEW RESULTS ON ORDERINGS ON SOFT SETS

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ABSTRACT
Molodtsov [2] introduced the soft set theory. Moreover Babitha and Sunil [5], [6] introduced partially ordered soft set. In this study we improve orderings on soft sets by giving new definitions such as filtered soft set, soft lattice, complete soft lattice, and some results related these definitions are studied.

Keywords: *Soft set,Partially ordered soft set, Soft lattice.*

ESNEK KÜMELERDE SIRALAMA ÜZERİNE BAZI YENİ SONUÇLAR

ÖZET
Molodtsov [2] esnek küme teorisi tanıtmıştır. Bununla birlikte Babitha ve Sunil [5], [6] kısmi sıralı esnek kümeleri tanıtmıştır. Biz bu çalışmada esnek süzgeç, esnek latis, tam esnek latis gibi yeni Tanımlar yaparak, bu tanımlar ile ilgili bazı sonuçlar vererek esnek kümelerdeki sıralamayı geliştirmeye çalıştık.

Anahtar Kelimeler: *Esnek küme, kısmi sıralı esnek küme, esnek latis.*

1. INTRODUCTION

Traditional methods can not be successfully used to solve complicated problems in economics, engineering and environment because of different kind of incomplete knowledge. For solving these complicated problems, Molodtsov [2] introduced the concept of soft sets in 1999. Maji et. al. [7] improved soft set theory by defining some structures, such as equality of two soft sets, subset and complement of a soft set, null soft set, and absolute soft set, intersection and union of soft sets.

Babitha [5] introduced the concepts of soft set relations as a sub soft set of the Cartesian product of the soft sets and discussed many related concepts such as equivalent soft set relation, partition, composition, function etc. Babitha and Sunil [6] defined ordering on a soft set and proved some set theoretical results based on ordering on soft set. Tanay and Yaylalı [1] gave some new structures for the partially ordered soft sets such as directed soft set, directed complete soft set and they introduced the soft Scott topology by using soft set relation. We refer to [4], [3], in order to refresh the fundamental concepts of set theory using in this paper. In this paper some new definitions such as filtered soft set, soft lattice, complete soft lattice for ordering on soft sets is given. Moreover, some results related these definitions are proved.

2. PRELIMINARIES AND BASIC DEFINITIONS

Definition 2.1. [2] Let U be an initial universe, E be the set of parameters. Let $P(U)$ be the set of all subsets of U and A be a subset of E. A pair (F, A) is called a soft set over U where $F: A \to P(U)$ is a set valued function.

Definition 2.2. [7] A soft set (F, A) over U is said to be a Null soft set denoted by Φ, if for every $\epsilon \in A$, $F(\epsilon) = \emptyset$

Definition 2.3. [7] For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) and denoted by $(F, A) \tilde{\subseteq} (G, B)$ if

(i) $A \subset B$ and,

(ii) $\forall \epsilon \in A$, $F(\epsilon)$ and $G(\epsilon)$ are identical approximations.

Definition 2.4. [5] Let (F, A) and (G, B) be two soft sets over U, then the cartesian product of (F, A) and (G, B) is defined as, $(F, A) \times (G, B) = (H, A \times B)$ where $H: A \times B \to P(U \times U)$ and $H(a, b) = F(a) \times G(b)$, where $(a, b) \in A \times B$
e.i. $H(a, b) = \{(h_i, h_j) | h_i \in F(a), h_j \in F(b)\}$

Definition 2.5. [5] Let (F, A) and (G, B) be two soft sets over U, then a relation R from (F, A) to (G, B) is a soft subset of $(F, A) \times (G, B)$. In other words, a relation R from (F, A) to (G, B) is of the form $R = (H_1, S)$ where $S \subset A \times B$ and $H_1(a, b) = H(a, b)$ for all $(a, b) \in S$ where $(H, A \times B) = (F, A) \times (G, B)$.

Definition 2.6. [5] Let R be a soft set relation from (F, A) to (G, B). Then the domain of R is defined as the soft set (D, A_1) where $A_1 = \{a \in A: H(a, b) \in for\ some\ b \in B\}$ and $D(a_1) = F(a_1), \forall a_1 \in A$.
The range of R is defined as the soft set (RG, B_1), where $B_1 \subset B$ and $B_1 = \{b \in B: H(a, b) \in R\ for\ some\ a \in A\}$ and $RG(b_1) = G(b_1), \forall b_1 \in B_1$.

Definition 2.7. [5] Let R be a relation on (F, A), then
i) R is reflexive: if $H_1(a, a) \in, \forall a \in A$.
ii) R is symmetric if $H_1(a, b) \in R \Rightarrow H_1(b, a)$.
iii) R is transitive if $H_1(a, b) \in R, H_1(b, c) \in R \Rightarrow H_1(a, c) \in R$ for every $a, b, c \in A$.

Definition 2.8. [6] A binary soft set relation R on (F, A) is an antisymmetric if $F(a) \times F(b) \in R$ and $F(b) \times F(a) \in R$ for every $F(a), F(b) \in (F, A)$ implies $F(a) = F(b)$.

Definition 2.9. [6] A binary soft set relation $\leq$ on (F, A) which is reflexive, antisymmetric and transitive is called a partial ordering of (F, A). The triple $(F, A, \leq)$ is called a partially ordered soft set.

Definition 2.10. [6] Let $\leq$ be an ordering of (F, A) and $F(a)$ and $F(b)$ be any two elements in (F, A). We say that $F(a)$ and $F(b)$ are comparable in the ordering if $F(a) \leq F(b)$ or $F(b) \leq F(a)$. We say that $F(a)$ and $F(b)$ are incomparable if they are not comparable.

Definition 2.11. [6] Let $\leq$ be a partially ordered on the soft set (F, A). Then $\leq$ is called a total or linear ordering on (F, A) if every element in (F, A) is comparable in the ordering $\leq$.

3. SOFT CHAIN, INFIMUM AND SUPREMUM OF SOFT SETS

Definition 3.1. [1] Let $B \subset A$ and $(G, B) \tilde{\subseteq} (F, A)$ where (F, A) is ordered by R. (G, B) is a soft chain in (F, A) if any two element in (G, B) are comparable.

Note that every soft subset (G, B) of a partially ordered soft set (F, A, R) is a partially ordered soft set with a soft set relation R. In fact, let $G(b)$ in (G, B) then $G(b) R G(b)$ since (F, A, R) is a partially ordered soft set. Then (G, B) is reflexive. Let $G(b_1) R G(b_2)$ and $G(b_2) R G(b_1)$. Since (F, A, R) is antisymmetric then $G(b_1) = G(b_2)$.Thus (G, B) is antisymmetric. Let $G(b_1) R G(b_2)$ and $G(b_2) R G(b_3)$. Since (F, A, R) is transitive then $G(b_1) R G(b_3)$. Thus (G, B) is transitive.

Definition 3.2. [6] Let $(G, B, \leq)$ be a partially ordered soft set. Then,
i) For $b \in B, G(b)$ is the least element of (G, B) in the ordering '$\leq$' if $G(b) \leq G(x)$ for all $x \in B$
ii) For $b \in B, G(b)$ is a minimal element of (G, B) in the ordering '$\leq$' if there exists no $x \in B$ such that $G(x) \leq G(b)$ and $G(x) \neq G(b)$.
i') For $b \in B, G(b)$ is the greatest element of (G, B) in the ordering '$\leq$' if for every $x \in B$, $G(x) \leq G(b)$.
ii') For $b \in B, G(b)$ is a maximal element of (G, B) in the ordering '$\leq$' if there exists no $x \in B$ such that $G(b) \leq G(x)$ and $G(x) \neq G(b)$.

Example 3.3. [1] Consider soft sets (F, A) and (G, B) as follows: The universe set $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and the parameter set $E = \{e_1, e_2, e_3, e_5, e_6\}$; $A = \{e_1, e_2, e_3, e_5, e_6\}$ and $B = \{e_1, e_3, e_5\}$ $F(e_1) = \{h_1, h_4\}$, $F(e_2) = \{h_1, h_3\}$, $F(e_3) = \{h_3, h_4, h_5\}$, $F(e_5) = \{h_1\}$, $F(e_6) = \{h_1, h_6\}$ and $G(e_1) = \{h_1, h_4\}$, $G(e_3) = \{h_3, h_4, h_5\}$, $G(e_5) = \{h_1\}$.
Hence $(G, B) \tilde{\subseteq} (F, A)$ Now define a soft set relation $\leq$ on (F, A) as $\leq = \{F(e_1) \times F(e_1), F(e_2) \times F(e_2), F(e_3) \times F(e_3), F(e_5) \times F(e_5), F(e_6) \times F(e_6), F(e_1) \times F(e_3), F(e_1) \times F(e_5), F(e_1) \times F(e_6), F(e_2) \times F(e_3), F(e_2) \times F(e_5), F(e_2) \times F(e_6), F(e_3) \times F(e_5), F(e_3) \times F(e_6), F(e_5) \times F(e_6)\}$ Then $F(e_1)$ and $F(e_2)$ are minimal elements of

(F,A); and $F(e_6)$ is the max- imal element and the greatest element of (F,A). Similarly since $G(e_1) \times G(e_1), G(e_3) \times G(e_3), G(e_5) \times G(e_5), G(e_1) \times G(e_3), G(e_3) \times G(e_5), G(e_1) \times G(e_5) \in \leq$ we can say $G(e_1)$ is a minimal element and the least element ; and also $G(e_5)$ is a maximal element and the greatest element of (G,B). Besides (G,B) is a soft chain in (F,A) because any two elements of (G,B) are comparable.

Theorem 3.4. Let (F,A) be ordered by $\leq$ and let $(G,B) \tilde{\subseteq} (F,A)$
a) (G,B) has at most one least element,
b) The least element of (G,B) (if it exists) is also minimal,
c) If (G,B) is a soft chain, the every minimal element of (G,B) is also least.
Proof a) Suppose there exists more than one least element in (G,B). Let $G(a)$ and $G(b)$ be two least elements, then by Definition 3.4. $G(a) = G(b)$.
b) Suppose $G(a)$ is the least element then $G(a) \leq G(x)$ for all $G(x)$ in (G,B). Hence there is no element $G(b)$ in (G,B) such that $G(b) \leq G(a)$ and $G(b) \neq G(a)$.Then $G(a)$ is a minimal.
c) Suppose (G,B) is a soft chain and $G(a)$ is a minimal element. Since (G,B) is a soft chain, $G(a)$ is comparable with $G(x)$ for all $x \in B$ and since $G(a)$ is a minimal element then $G(a) \leq G(x)$ for all $x \in B$. Hence $G(a)$ is the least element.
Note that one can replace in Theorem 3.4. the words ”least” and ”minimal” by the words ”greatest” and ”maximal”.

Definition 3.5. [1] Let $\leq$ be an ordering of (F,A), let $(G,B) \tilde{\subseteq} (F,A)$.
i) For $a \in A, F(a)$ is a lower bound of (G,B) in the ordered soft set $(F,A,\leq)$ if $F(a) \leq G(x)$ for all $x \in B$.
ii) For $a \in A, F(a)$ is called infimum of (G,B) in $(F,A,\leq)$ (or the greatest lower bound) if it is the greatest element of the soft set of all lower bounds of (G,B) in $(F,A,\leq)$.
Similarly,
i') For $a \in A, F(a)$ is an upper bound of (G,B) in the ordered soft set $(F,A,\leq)$ if $G(x) \leq F(a)$ for all $x \in B$.
ii') For $a \in A, F(a)$ is called supremum of (G,B) in $(F,A,\leq)$ (or the least upper bound) if it is the least element of the soft set of all upper bounds of (G,B) in $(F,A,\leq)$.

Example 3.6. [1] Consider the soft sets (F,A) and (G,B) as presented in Example 3.3.. $F(e_5)$ and $F(e_6)$ are upper bounds of (G,B) in the ordered soft set $(F,A,\leq)$. $F(e_5)$ is the supremum of (G,B). $F(e_1)$ is the lower bound of (G,B) and so it is the infimum of (G,B) in $(F,A,\leq)$.

Theorem 3.7. Let $(F,A,\leq)$ be an ordered soft set and let $(G,B) \tilde{\subseteq} (F,A)$
a) (G,B) has at most one infimum.
b) If $G(b)$ is the least element of (G,B), then $G(b)$ is the infimum of (G,B).
c) If $F(b) = \inf(G,B)$ and $b \in B$, then $F(b)$ is the least element of (G,B).
d) For $b \in A, F(b)$ is an infimum of (G,B) in $(G,B) \tilde{\subseteq} (F,A)$ iff
    i) $F(b) \leq F(x)$ for all $x \in B$.
    ii) If $F(b') \leq F(x)$ for all $x \in B$ then $F(b') \leq F(b)$.
Proof a) Since infimum is the greatest element of the soft set of all lower bounds of (G,B) and since the greatest element is at most one, (G,B) has at most one greatest element.
b) Suppose $G(b)$ is the least element of (G,B). If $F(a)$ is a lower bound of (G,B), $F(a) \leq G(b)$ because $G(b)$ is an element of (G,B).So $G(b)$ is the greatest lower bound of (G,B). Hence $G(b)$ is infimum of (G,B).
c) Suppose $F(b) = \inf(G,B)$ and $b \in B$. Then $F(b) \leq F(x)$ for all $x \in B$. Since $b \in B$ $F(b)$ is an element of (G,B). Therefore $F(b)$ is the least element of (G,B).
d) By definition of infimum.
Note that, the Theorem 3.7. a, b, c remains true if the words ”least” and ”infimum” replaced by the words ”greatest” and ”supremum”.
Also in Theorem 3.7. d $F(b)$ is supremum of (G,B) in $(G,B) \tilde{\subseteq} (F,A)$ for $b \in A$, iff
    i) $F(x) \leq F(b)$ for all $x \in B$.
    ii) ii) If $F(x) \leq F(b')$ for all $x \in B$ then $F(b) \leq F(b')$.

4. DIRECTED, FILTERED SOFT SETS AND SOFT SET LATTICES

Definition 4.1. [1] Consider a soft set (F,A) equipped with reflexsive, transitive relation $\leq$. This soft set relation is called preorder and (F,A) is a preordered soft set.

Definition 4.2. Let (F,A) be a soft set. (F,A) is called a finite soft set, if it is a soft set with a finite parameter set.

Definition 4.3. [1] Let (F, A) be a preordered soft set. A soft subset (G, B) of (F, A) is directed provided it is nonnull and every finite soft subset of (G, B) has an upperbound in (G, B).

Definition 4.4. Let (F, A) be a preordered soft set. We call a nonnull soft subset (G, B) of (F, A) filtered if every finite soft subset of (G, B) has a lower- bound.

Example 4.5. Consider the soft set (F, A) over U, where $U = \{c_1, c_2, c_3, c_4, c_5, c_6\}$, $A = \{a_1, a_2, a_3\}$ and $F(a_1) = \{c_1, c_2\}$, $F(a_2) = \{c_2\}$, $F(a_3) = \{c_4, c_5, c_6\}$ and $\leq = \{F(a_1) \times F(a_1), F(a_2) \times F(a_2), F(a_3) \times F(a_3), F(a_1) \times F(a_2), F(a_2) \times F(a_3), F(a_1) \times F(a_3)\}$. (F, A) is directed and filtered soft set with the soft set relation $\leq$.

Example 4.6. Consider the soft set (F, A) as presented in Example 4.5.. Let $\leq' = \{F(a_1) \times F(a_1), F(a_2) \times F(a_2), F(a_3) \times F(a_3), F(a_1) \times F(a_3)\}$. (F, A) with the soft set relation $\leq'$ is neither directed and nor filtered since (F, A) itself is finite but it does not have upperbound.

Definition 4.7. Let (F, A) be a soft set with a preorder $\leq$. For $(G, B) \tilde{\subseteq} (F, A)$
i) [1] $\downarrow (G, B) = (H, C)$ where $C = \{a \in A: F(a) \leq G(b) for\ some\ b \in B\}$ and $H = F|_C$.
ii) [1] $\uparrow (G, B) = (K, D)$ where $D = \{a \in A: G(b) \leq F(a) for\ some\ b \in B\}$ and $K = F|_D$.
We also say
iii) [1] (G, B) is a lower soft set iff $(G, B) = \downarrow (G, B)$.
iv) [1] (G, B) is an upper soft set iff $(G, B) = \uparrow (G, B)$.
v) [1] (G, B) is an soft ideal iff it is a directed lower soft set.
vi) (G, B) is a soft filter iff it is a filtered upper soft set.
vii) A soft ideal is principal iff it has a maximum element.
viii) A soft filter is principal iff it has a minimum element.
ix) $Id(F, A)$ is the set of all ideals of (F, A).
x) $Filt(F, A)$ is the set of all filters (F, A).
xi) $Id_0(F, A) = Id(F, A) \tilde{\cup} \{\Phi\}$
xii) $Filt_0(F, A) = Filt(F, A) \tilde{\cup} \{\Phi\}$

Example 4.8. [1] Consider the soft set (F, A) and the soft set relation R as presented in Example 4.5.. Let $B = \{a_1, a_2\}$ and $G: B \to P(U)$ such that $G(a_1) = \{c_1, c_2\}$, $G(a_2) = \{c_2\}$. Thus $(G, B) \tilde{\subseteq} (F, A)$.
$\uparrow (G, B) = (H, C)$ where $C = \{a \in A: G(b) R F(a)\ for\ some\ b \in B\} = \{a_1, a_2, a_3\}$ and $H = F|_C$. $\downarrow (G, B) = (K, D)$ where $D = \{a \in A: F(a) \leq G(b)\ for\ some\ b \in B\} = \{a_1, a_2\}$ and $K = F|_D$.
Since $D = B$, we have $\downarrow (G, B) = (G, B)$. Hence (G, B) is a lower soft set.

Definition 4.9. A soft inf-semilattice is a partially ordered soft set $(F, A, \leq)$ in which $F(a), F(b)$ have infimum for any two elements $a, b \in A$. A soft sup-semilattice is a partially ordered soft set (F, A) in which any $F(a), F(b)$ in (F, A) have a supremum. A partially ordered soft set (F, A) which is both soft inf-semilattice and soft sup-semilattice is called a soft lattice.

Definition 4.10. Let (F, A) be a soft semilattice. A soft subset (G, B) of (F, A) is called soft subsemilattice of (F, A) if (G, B) is soft semilattice in (F, A).

Theorem 4.11. In soft inf-semilattice an upper soft set is a filter iff it is a soft subsemilattice. A dual holds for lower soft sets and ideals in soft sup- semilattice.
Proof ($\Rightarrow$) Suppose (F, A) is a soft inf-semilattice and any (U, B) upper soft set is a filter in (F, A). Take $U(a)$ and $U(b)$ in (U, B), then $U(a), U(b)$ have a lower bound $U(c)$ in (U, B) since (U, B) is a filter. Thus (U, B) is a soft inf-semilattice in (F, A).
($\Leftarrow$) Let $(U, B) = \downarrow (U, B)$. Since (U, B) is a soft subsemilattice of a soft inf- semilattice (F, A), for all finite soft subset of (U, B), there exists an infimum in (U, B). Thus (U, B) is a filter.

Definition 4.12. i) [1] A posset is said to be directed complete soft sets if every directed soft subset has a supremum.
ii) A posset which is an soft inf-semilattice and directed complete will be called a directed complete soft inf-semilattice.
iii) A soft lattice is called complete soft lattice in which every soft subset has a supremum and infimum. A totally ordered complete soft lattice is called a complete soft chain.
iv) A posset is called a complete soft inf-semilattice iff every nonnull soft subset has an infimum and every directed soft subset has a supremum.

v) A posset is called bounded complete, if every soft subset that is bounded above has a least upper bound.

Example 4.13. Take $A = \mathbb{Z}^+$ and let $U = \mathbb{R}^*$ extended real numbers and $F: \mathbb{Z}^+ \to P(\mathbb{R}^*)$ such that $F(a) = (1, a+1]$ for $a \in A$. So (F, A) is a soft set. It is a partially ordered soft set by inclusion. Also it is soft lattice, because finite soft subsets have supremum and infimum; such that $(1, a+1] \subseteq (1, b+1]$ whenever $a \leq b$. Furthermore (F, A) is directed complete soft set since every directed soft subset has a supremum, directed complete inf-semilattice, since it is directed complete and soft inf-semilattice, complete inf-semilattice, and bounded complete soft set.

Theorem 4.14. Let (F, A) be a partially ordered soft set. Then,
a) If arbitrary soft set has a supremum, it has also an infimum.
b) If arbitrary soft subsets of (F, A) has supremum (or infimum), then (F, A)
is a complete soft lattice.
Proof a) Let $(G, B) \tilde{\subseteq} (F, A)$ and let $(H, C) = \tilde{\cap} \{\downarrow G(b) | b \in B\}$ be the set of lower bounds of (G, B) (If (G, B) is a null soft set we take $(H, C) = (F, A)$). Now we ned to show $\sup(H, C) = \inf(G, B)$. If $b \in B, G(b)$ is an upper bound of (H, C); whence $\sup(H, C) \leq G(b)$. This proves that $\sup(H, C) \in (H, C)$; as it clearly is the maximal element of B, this also proves that X has a greatest lower bound. (There is obviously a dual argument assuming infimums exist.)
b) It is straightforward from (1).

5. CONCLUSION

In this paper, we gave definition of soft lattice and introduced the completeness for the soft sets were introduced. Also some examples were given and some results were proved. One can use this paper, to improve the notion of ordering on soft sets and to construct a relation between orderings and soft set function.

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